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Isolating vacuum amplitudes in quantum field calculations at finite temperature

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Abstract

In calculating Feynman diagrams at finite temperature, it is sometimes convenient to isolate subdiagrams which do not depend explicitly on the temperature. We show that, in the imaginary time formalism, such a separation can be achieved easily by exploiting a simple method, due to M. Gaudin, to perform the sum over the Matsubara frequencies. In order to manipulate freely contributions which may be individually singular, a regularization has to be introduced. We show that, in some cases, it is possible to choose this regularization in such a way that the isolated subdiagrams can be identified with analytical continuations of vacuum n-point functions. As an aside illustration of Gaudin's method, we use it to prove the main part of a recent conjecture concerning the relation which exists in the imaginary time formalism between the expressions of a Feynman diagram at zero and finite temperature.

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I. INTRODUCTION

In doing quantum field theory calculations at finite temperature [1, 2], it is often useful to separate contributions of subdiagrams which do not explicitly depend on the temperature. In some cases, these subdiagrams can be identified with amplitudes calculated with the rules of zero temperature, and we shall refer to them as vacuum amplitudes. Isolating such amplitudes is useful in particular to analyze the ultraviolet divergences. These divergences are associated with the short distance singularities in the propagator, and these are not modified by the temperature [2, 3]. Thus, one expects the ultraviolet divergences to be those of the vacuum subdiagrams.

Identifying vacuum amplitudes in an arbitrary Feynman diagram at finite temperature is, in general, not an easy task. In the imaginary time formalism, which is the one we shall use in this paper, the calculation of a Feynman diagram at finite temperature is similar to the corresponding calculation at zero temperature, the integrals over energies being replaced by sums over the discrete Matsubara frequencies. (Note also that the sums over Matsubara frequencies go over the Euclidean integrals of field theory in the limit of zero temperature.) In this formalism, the temperature dependence appears explicitly, after the sums over Matsubara frequencies have been done, through statistical factors which vanish when the temperature vanishes. This makes it easy, in principle, to identify the zero temperature contributions. This procedure works well for one loop-diagrams which are linear in the statistical factors, but things become more subtle in higher loop orders. There is indeed a further complication. Depending on how one proceeds to perform the calculation, one may end up with expressions containing different numbers of statistical factors, or statistical factors whose arguments involve energies attached to different lines of the diagram. This makes the separation of various contributions difficult.

There is however a method to perform the sums over Matsubara frequencies which leads directly to a result where the number of statistical factors is related to the number of loops, and the arguments of the statistical factors are energies attached to single lines of the diagram. This method was developed by M. Gaudin a long time ago [4], but seems to have been largely ignored in the recent literature. We shall use this method to analyze the separation of vacuum and thermal contributions in a general Feynman diagram. The main problem in doing the sum over Matsubara frequencies is the choice of the independent

frequencies. In his work, Gaudin relates the various choices of independent variables to the various trees that one can build with the lines of the diagram [4]. Note that, more recently [5], tree diagrams were also used in a similar context, but the general rules that apply to arbitrary loop order were not given.

We shall show that the rules proposed by Gaudin allow us to organize the result of a calculation of a Feynman diagram at finite temperature according to powers of statistical factors. In some cases, this will allow us to isolate vacuum amplitudes, that is vacuum subdiagrams that one can relate to analytic continuations of Euclidean vacuum amplitudes. This connection is not always possible to realize however. This is because, at some intermediate step of the analysis, one needs to introduce a regularization to give meaning to otherwise singular individual contributions (whose sum is regular). For the regularization that we use, we can find counter-examples of vacuum subdiagrams which are not analytic continuations of corresponding, i.e., topologically identical, vacuum Euclidean amplitudes.

The paper is organized as follows. In the next section, we consider several examples which illustrate various features of calculations of Feynman diagrams at finite temperature. These examples allow us to specify concretely what is involved in identifying vacuum amplitudes. They also serve as a pedagogical introduction for the general method of summing over Matsubara frequencies that is discussed in Sect. III. The rules presented in Sect. III are essentially those derived by M. Gaudin [4]. They are used in Sect. IV in order to write down an expansion of the temperature dependent pieces of a Feynman diagram in terms of powers of statistical factors (which vanish as the temperature vanishes). We show that this decomposition exists only if some regularization is introduced to control individually singular terms. In some cases we can relate the vacuum subdiagrams to well defined analytic continuations of Euclidean amplitudes. But it is not always possible to do so, as illustrated by a counter-example that we present at the end of Sect. IV. Conclusions are presented in Sect. V. Finally, in the appendix, we show how the main part of the conjecture of Ref. [6] follows directly form the rules of Sect. III.

II. SIMPLE EXAMPLES

In this section, we work out simple examples of finite temperature calculations in scalar field theory. Our goal, beyond introducing the basic notation, is to illustrate on a few cases

how one can isolate, in a given Feynman diagram, a contribution to either the full diagram or to a subdiagram, which does not explicitly depend on the temperature. In the cases studied here, such contributions can be associated to analytic continuations of vacuum n-point functions. As we proceed, we shall also recall some elementary techniques to perform sums over Matsubara frequencies which will be generalized in the next section.

A. General definitions

In the imaginary time formalism, the (time-ordered) propagator of a free scalar field can be written in a mixed representation, as a function of momentum $\mathbf{p}$ and imaginary time τ, as follows [2]:

$$\begin{aligned} D_0(\tau_1-\tau_2,\mathbf{p}) &= \int \mathrm{d}^3x\ \mathrm{e}^{-i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)}\ \langle \mathrm{T}\phi(\tau_1,\mathbf{x}_1)\phi(\tau_2,\mathbf{x}_2)\rangle \\ &= \frac{1}{2\varepsilon_p}\left\{(1+n_{\varepsilon_p})e^{-\varepsilon_p|\tau_1-\tau_2|}+n_{\varepsilon_p}e^{\varepsilon_p|\tau_1-\tau_2|}\right\}. \end{aligned} \tag{1}$$

In this expression, valid for $|\tau_1-\tau_2|\leq\beta=1/T$, with T the temperature, n_{ε_p} is the Bose-Einstein statistical factor:

$$n_\varepsilon=\frac{1}{\mathrm{e}^{\beta\varepsilon}-1},\qquad n_{-\varepsilon}=-1-n_\varepsilon, \tag{2}$$

and $\varepsilon_p=\sqrt{p^2+m^2}$ is the energy of a mode with momentum p. We shall write the statistical factor indifferently as n_ε or $n(\varepsilon)$. The propagator in Eq. (1) can be used in perturbative calculations at finite temperature. The calculation of a diagram Γ proceeds then typically as follows: After having chosen an orientation [11] for each line of Γ, one associates to each vertex v_i of Γ an imaginary time τ_i, and to a line joining the vertex v_i to the vertex v_j one associates the propagator $D_0(\tau_j-\tau_i,\mathbf{p})$. The contribution of the diagram is then obtained by integrating over all the time variables τ_i between 0 and β. We shall give soon an example of such a calculation. For simple diagrams, this technique can be quite convenient, and indeed it has been used in a systematic analysis of the one loop contributions in QCD at finite temperature [7]. However, because the propagators take different forms according to the sign of $\tau_j-\tau_i$, one needs to treat separately the various integration subdomains, and this becomes rapidly cumbersome for high order diagrams.

An alternative is to use an energy (or frequency) representation of the propagator, which

we shall write in the form (ω is a complex variable):

$$D(\omega, \mathbf{p}) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}p_0}{2\pi} \frac{\rho(p_0, \mathbf{p})}{p_0 - \omega}, \tag{3}$$

where the quantity $\rho(p_0, \mathbf{p})$ is the spectral function. It is an odd function of p_0. In most of the arguments of this paper, the spectral function needs not be that of free particles. We call ρ_0 and D_0, respectively, the spectral function and the propagator for free particles:

$$\rho_0(p_0, \mathbf{p}) = \frac{\pi}{\varepsilon_p} \left[\delta(p_0 - \varepsilon_p) - \delta(p_0 + \varepsilon_p)\right], \qquad D_0(\omega, \mathbf{p}) = \frac{1}{\varepsilon_p^2 - \omega^2}. \tag{4}$$

The propagator (3) is an analytic function of ω everywhere in the complex ω-plane, except on the real axis; above (below) the real axis it coincides with the retarded (advanced) propagator. The imaginary time propagator (1) can be recovered from the Fourier coefficients $D(i\omega_n, \mathbf{p})$, where $\omega_n = 2\pi nT$ is a Matsubara frequency:

$$D(\tau, \mathbf{p}) = \frac{1}{\beta} \sum_n \mathrm{e}^{i\omega_n \tau} D(i\omega_n, \mathbf{p}). \tag{5}$$

While the sum in Eq. (5) can be done easily for free particles by using the expression of $D_0(\omega, \mathbf{p})$ given above, Eq. (4), we shall often perform the sum over Matsubara frequencies after using the spectral representation of the propagator, Eq. (3). Then the following formula will be useful:

$$\frac{1}{\beta} \sum_n \frac{\mathrm{e}^{i\omega_n \tau}}{p_0 - i\omega_n} = \epsilon_\tau \, n(\epsilon_\tau \, p_0) \, \mathrm{e}^{p_0 \tau}, \tag{6}$$

where $\epsilon_\tau = 1$ if $\tau > 0$ and $\epsilon_\tau = -1$ if $\tau < 0$. It follows that:

$$D(\tau, \mathbf{p}) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}p_0}{2\pi} \mathrm{e}^{p_0 \tau} \rho(p_0, \mathbf{p}) \, \epsilon_\tau n(\epsilon_\tau p_0). \tag{7}$$

The r.h.s. of Eq. (7) provides two equivalent ways to calculate $D(\tau = 0, \mathbf{p})$, by letting τ go to 0 by positive or negative values. Both are of course equivalent. (Note that $D(\tau, \mathbf{p})$ is not analytic at $\tau = 0$: it is continuous, but it's derivative is not.)

The evaluation of the fluctuations of the free scalar field provides the simplest example where the separation of vacuum and thermal contributions can be realized easily. Note that since the statistical factors are explicit in the expression (1) of the propagator, using this expression makes it straightforward to separate these contributions. Indeed, by using Eq. (1) one obtains immediately:

$$\langle \phi^2 \rangle = \int \frac{d^3p}{(2\pi)^3} D_0(\tau = 0, \mathbf{p}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\varepsilon_p} \left(1 + 2n_{\varepsilon_p}\right) \equiv I_0 + I_1, \tag{8}$$

where the two terms in the right hand side are the zero temperature contribution I_0 and the finite temperature one I_1.

Alternatively, we can use the Fourier representation, Eq. (5), and Eq. (7), in order to write $D_0(\tau=0,\mathbf{p})$ as:

$$\int_{-\infty}^{+\infty}\frac{\mathrm{d}p_0}{2\pi}\rho(p_0,\mathbf{p})n(p_0)=-\int_{-\infty}^{+\infty}\frac{\mathrm{d}p_0}{2\pi}\rho(p_0,\mathbf{p})\theta(-p_0)+\int_{-\infty}^{+\infty}\frac{\mathrm{d}p_0}{2\pi}\rho(p_0,\mathbf{p})\epsilon(p_0)n(|p_0|). \quad (9)$$

Now the separation of the thermal contribution is achieved with the help of the formula:

$$n(p_0)=-\theta(-p_0)+\epsilon(p_0)n(|p_0|), \quad (10)$$

where $\epsilon(p_0)=1$ or $\epsilon(p_0)=-1$ depending on the sign of p_0 ($\epsilon(p_0)=\theta(p_0)-\theta(-p_0)$). As $T\to 0$, $n(|p_0|)\to 0$ and $n(p_0)\to -\theta(-p_0)$. Note that since $\rho(p_0)$ is an odd function we have:

$$-\int_{-\infty}^{+\infty}\frac{\mathrm{d}p_0}{2\pi}\rho(p_0,\mathbf{p})\theta(-p_0)=\int_{-\infty}^{+\infty}\frac{\mathrm{d}p_0}{2\pi}\rho(p_0,\mathbf{p})\theta(p_0). \quad (11)$$

This gives us two possibilities to write the zero temperature piece in Eq. (9), either as $-\theta(-p_0)$ (as we have done in Eq. (9)), or as $\theta(p_0)$. These two choices are of course closely related to the two ways in which one can let τ go to zero to get $D(\tau=0,\mathbf{p})$ from Eq. (7). As for the last term in Eq. (9), it is useful to note that it is unaffected by the change in the sign of p_0.

Because I_0 is ultraviolet divergent, it may be convenient to write it as an Euclidean integral:

$$I_0=\int\frac{d^4p}{(2\pi)^4}D(ip_0,\mathbf{p}). \quad (12)$$

This allows us in particular to use covariant regulators to evaluate it.

The finite temperature contribution I_1 is defined in Eq. (8). It can also be obtained from the last term in Eq. (9), which allows us to write it also as a 4-dimensional integral, which will prove convenient in our forthcoming analysis. We shall introduce a special notation for the integrand in this last term of Eq. (9):

$$\sigma(p_0,\mathbf{p})\equiv\rho(p_0,\mathbf{p})\epsilon(p_0)n(|p_0|). \quad (13)$$

This particular combination of the spectral density and the statistical factor will appear systematically in the calculations. The function $\sigma(p_0,p)$ is an even, positive, function of p_0. For free particles, $\sigma=\sigma_0$, with:

$$\sigma_0(p_0,\mathbf{p})=\frac{\pi}{\varepsilon_p}\left[\delta(p_0-\varepsilon_p)+\delta(p_0+\varepsilon_p)\right]n_{\varepsilon_p}. \quad (14)$$

In terms of σ the thermal contribution to $\langle\phi^2\rangle$ is simply:

$$I_1 = \int \frac{d^4p}{(2\pi)^4}\,\sigma(p_0, \mathbf{p}). \tag{15}$$

B. One loop in two ways

We now proceed to the analysis of our first non trivial example, that of the one loop contribution to the self-energy in a ϕ^3 scalar theory. For this example, we shall present two calculations, one using the time representation, one using the frequency representation. In both cases, we shall focus on the separation of the contribution which depends on the temperature from that which does not.

1. *Time representation*

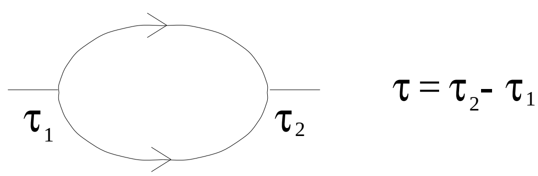


FIG. 1: One-loop self-energy in ϕ^3 scalar field theory, time representation

We start by writing the diagram in the time representation (see Fig. 1). For the particular choice of orientation given in Fig. 1 (the final result is independent of the specific choice), the diagram contributes the following integral:

$$I(\tau, \mathbf{k}) = \int \frac{d^3p}{(2\pi)^3} D_0(\tau, \mathbf{p}) D_0(\tau, \mathbf{k} - \mathbf{p}). \tag{16}$$

The Fourier transform of $I(\tau, \mathbf{k})$ is obtained through the formula:

$$I(i\omega_e, \mathbf{k}) = \int_0^\beta d\tau\, e^{-i\omega_e \tau} I(\tau, \mathbf{k}), \tag{17}$$

where ω_e is an external Matsubara frequency. By using the explicit form of the propagator given in Eq.(1), we obtain (with $\mathbf{q} = \mathbf{k} - \mathbf{p}$):

$$\begin{aligned} I(i\omega_e, \mathbf{k}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\varepsilon_p} \frac{1}{2\varepsilon_q} \Bigg\{ &(1 + n_{\varepsilon_p} + n_{\varepsilon_q}) \left(\frac{1}{i\omega_e + \varepsilon_p + \varepsilon_q} - \frac{1}{i\omega_e - \varepsilon_p - \varepsilon_q} \right) \\ &+ (n_{\varepsilon_p} - n_{\varepsilon_q}) \left(\frac{1}{i\omega_e - \varepsilon_p + \varepsilon_q} - \frac{1}{i\omega_e + \varepsilon_p - \varepsilon_q} \right) \Bigg\}. \end{aligned} \tag{18}$$

In order to arrive at this expression, we have used the fact that ω_e is a Matsubara frequency so that $\mathrm{e}^{i\beta\omega_e} = 1$. We have also used simple identities like $1+n_{\varepsilon_k} = e^{\beta\varepsilon_k} n_{\varepsilon_k}$ in order to eliminate the exponential factors resulting from the τ integration. Note that terms containing products of statistical factors, which appear in the initial stage of the calculation, have cancelled out in the final formula.

The expression (18) exhibits the contributions of the various physical processes that take place in the heat bath: pair creation or annihilation, which occur also in the vacuum, and scattering processes which take place only in the heat bath.

The separation of vacuum and thermal contributions in Eq.(18) is straightforward, and proceeds in the same way as for the fluctuation calculation above (see Eq. (8)). The vacuum part is obtained by dropping in Eq. (18) the terms which contain a statistical factor:

$$I^{(0)}(i\omega_e, \mathbf{k}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\varepsilon_p} \frac{1}{2\varepsilon_q} \left(\frac{1}{i\omega_e + \varepsilon_p + \varepsilon_q} - \frac{1}{i\omega_e - \varepsilon_p - \varepsilon_q} \right). \tag{19}$$

The thermal contribution, the sum of terms with one statistical factor, can be put in a compact form by exploiting the symmetries of the diagram in regrouping terms. One easily gets:

$$I^{(1)}(i\omega_e, \mathbf{k}) = 2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\varepsilon_p} n_{\varepsilon_p} \left[D_0(i\omega_e - \varepsilon_p, \mathbf{k} - \mathbf{p}) + D_0(i\omega_e + \varepsilon_p, \mathbf{k} - \mathbf{p}) \right], \tag{20}$$

where we have used the expression (4) of $D_0(\omega)$.

Note that in the calculations above it was essential to keep ω_e as a Matsubara frequency (so that e.g. $\mathrm{e}^{i\beta\omega_e} = 1$). The final formula however gives I as an analytic function which can be continued to all values of $\omega = i\omega_e$ in the complex plane, with singularities on the real axis.

2. *Frequency representation*

Alternatively, one may start from the energy representation and label each line with independent Matsubara frequencies ω_m and ω_n (see Fig. 2). Taking into account the energy conservation at the vertices, we may set $\omega_m = \omega_e - \omega_n$, and obtain:

$$I(i\omega_e, \mathbf{k}) = \frac{1}{\beta} \sum_n \int \frac{d^3p}{(2\pi)^3} D(i\omega_n, \mathbf{p}) D(i\omega_e - i\omega_n, \mathbf{k} - \mathbf{p}). \tag{21}$$

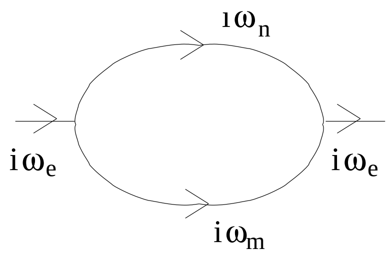


FIG. 2: One-loop self-energy in ϕ^3 scalar field theory, frequency representation.

Next, we use the spectral representation (3) for each propagator and perfom the sum over the Matsubara frequency ω_n. One gets ($\mathbf{q} = \mathbf{k} - \mathbf{p}$):

$$I(i\omega_e, \mathbf{k}) = \int \frac{d^3p}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \rho(p_0, \mathbf{p}) \int_{-\infty}^{\infty} \frac{dq_0}{2\pi} \rho(q_0, \mathbf{q}) \frac{n_{q_0} - n_{-p_0}}{p_0 + q_0 - i\omega_e}. \tag{22}$$

Note that the numerator in Eq. (22) can be written also as $n_{p_0} - n_{-q_0}$ ($= n_{q_0} - n_{-p_0}$). At this point, we can use Eq. (3) to perform trivially one of the energy integrals in Eq. (22). One gets then:

$$\begin{aligned} I(i\omega_e, \mathbf{k}) &= \int \frac{d^3p}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \rho(p_0, \mathbf{p}) [-n(-p_0)] D(i\omega_e - p_0, \mathbf{k} - \mathbf{p}) \\ &+ \int \frac{d^3p}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dq_0}{2\pi} \rho(q_0, \mathbf{k} - \mathbf{p}) [n(q_0)] D(i\omega_e - q_0, \mathbf{p}). \end{aligned} \tag{23}$$

The separation of the finite temperature contribution is now easily achieved with the help of Eq. (10). A simple calculation allows us to recover Eq. (19) above for $I^{(0)}$. Note that $I^{(0)}$ can also be written as a 4-dimensional Euclidean integral:

$$I^{(0)}(i\omega_e, \mathbf{k}) = \int \frac{d^4p}{(2\pi)^4} D(ip_0, \mathbf{p}) D(i\omega_e - ip_0, \mathbf{k} - \mathbf{p}). \tag{24}$$

This is obvious from Eq. (21), but can be also verified directly starting from Eq. (23). We shall often use the following notation for integrals such as that in Eq. (24):

$$I^{(0)}(K) = \int \frac{d^4P}{(2\pi)^4} D(P) D(K - P), \tag{25}$$

with $K = (i\omega_e, \mathbf{k})$, $P = (ip_0, \mathbf{p})$.

As for I_1, we shall write it as:

$$I^{(1)}(i\omega_e, \mathbf{k}) = 2 \int \frac{d^4p}{(2\pi)^4} \sigma(p_0, \mathbf{p}) D(i\omega_e - p_0, \mathbf{k} - \mathbf{p}), \tag{26}$$

where $\sigma(p_0, \mathbf{p})$ is defined in Eq. (13). A simple calculation, using the free spectral function allows us to recover Eq. (20). We shall argue later that the formulae (23) and (26) can be written directly, i.e., without calculation, by using an appropriate set of rules.

In view of future developments, we redo now the sum over Matsubara frequencies in a way that may look at this point artificially complicated. Let us keep the labels ω_n and ω_m of the two lines as they are in Fig. 2, and write the resulting product of denominators in the integral (21) as:

$$\frac{1}{p_0 - i\omega_n} \frac{1}{q_0 - i\omega_m} = \frac{1}{p_0 + q_0 - i\omega_e} \left\{ \frac{1}{p_0 - i\omega_n} + \frac{1}{q_0 - i\omega_m} \right\}. \tag{27}$$

This equation is valid if $\omega_n + \omega_m = \omega_e$ (which implies in particular that ω_e is a Matsubara frequency at this stage of the calculation). But on the right hand side the sum over the Matsubara frequencies can be done independently on ω_n and ω_m. There is however one subtlety related to the fact that these individual sums are ill-defined. Following Gaudin [4], we introduce a regulator and rewrite Eq. (27) as

$$\frac{\mathrm{e}^{i\omega_n \tau_n}}{p_0 - i\omega_n} \frac{\mathrm{e}^{i\omega_m \tau_m}}{q_0 - i\omega_m} = \frac{1}{p_0 + q_0 - i\omega_e} \left\{ \frac{\mathrm{e}^{i\omega_e \tau_m}\, \mathrm{e}^{i\omega_n(\tau_n - \tau_m)}}{p_0 - i\omega_n} + \frac{\mathrm{e}^{i\omega_e \tau_n}\, \mathrm{e}^{i\omega_m(\tau_m - \tau_n)}}{q_0 - i\omega_m} \right\}, \tag{28}$$

where in the right hand side we have used the relation $\omega_n + \omega_m = \omega_e$ to express each term as a function of the Matsubara frequency which sits in the denominator. The sum over Matsubara frequencies are now well defined: in the left hand side, it is so even in the absence of the regulator, and we can let τ_n and τ_m go to zero; in the right hand side, the sums are well defined provided we keep $\tau_n - \tau_m \neq 0$ when we take the limit. Using the formula (6) to perform the sums, we then easily recover Eq. (22) with the two ways of writing the numerator corresponding to the two possible limits $\tau_n - \tau_m \to 0\pm$. This method will be generalized in our next example.

C. Two loop in one way

Our next example is the 2-loop contribution to the free energy in ϕ^3 scalar field theory displayed in Fig. 3. While a calculation through integrations in the imaginary time could be done as easily as in the previous case (there is again only one integration to be done), we go here directly to the frequency representation and proceed by labelling the various internal lines of the diagram with independent Matsubara frequencies.

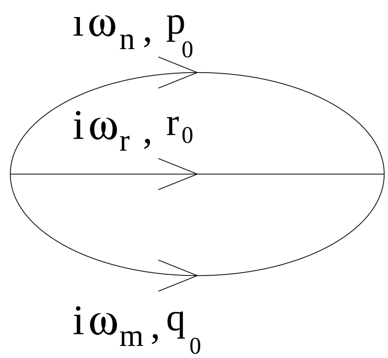


FIG. 3: A two-loop contribution to the pressure in scalar ϕ^3 theory. To each line are attached two labels: a Matsubara frequency and a real variable representing the energy variable of the spectral function of the propagator.

First, we take into account the conservation of energy in order to work with independent frequencies, i.e., we set $\omega_r = -\omega_n - \omega_m$. One is then led to calculate the following sum-integral (we set $\mathbf{r} = \mathbf{k} - \mathbf{p} - \mathbf{q}$):

$$I = \frac{1}{\beta^2}\sum_n\sum_m \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3} D(i\omega_n,\mathbf{p})D(i\omega_m,\mathbf{q})D(-i\omega_n - i\omega_m,\mathbf{r}). \tag{29}$$

Using as before the spectral representation for each propagator, one ends up with the following sum over Matsubara frequencies:

$$\frac{1}{\beta^2}\sum_{n,m}\frac{1}{(p_0 - i\omega_n)(q_0 - i\omega_m)(r_0 + i\omega_n + i\omega_m)} = \frac{(-n_{q_0} + n_{-r_0})\,(-n_{p_0} + n_{-q_0-r_0})}{r_0 + p_0 + q_0}. \tag{30}$$

In order to obtain the right hand side, we have performed the sum over ω_m first and then that over ω_n. The statistical factor $n_{-q_0-r_0}$ involving a linear combination of frequencies can be transformed using the identity:

$$(-n_{q_0} + n_{-r_0})\,n_{-q_0-r_0} = (1 + n_{-q_0} + n_{-r_0})\,n_{-q_0-r_0} = n_{-q_0}n_{-r_0}. \tag{31}$$

One then obtains:

$$\frac{1}{\beta^2}\sum_{n,m}\frac{1}{(p_0 - i\omega_n)(q_0 - i\omega_m)(r_0 + i\omega_n + i\omega_m)} = \frac{n_{p_0}n_{q_0} - n_{p_0}n_{-r_0} + n_{-q_0}n_{-r_0}}{r_0 + p_0 + q_0}. \tag{32}$$

There is a more systematic way to arrive directly at the expression (32) in which each statistical factor is function of the energy variable carried by a single line (rather than involving sums of energy variables as in Eq. (30)). This requires leaving open the choice of the independent Matsubara frequencies so as to allow the calculation to proceed in as a symmetrical

way as possible. Let us then attach to the three internal lines the variables $\{p_0, q_0, r_0\}$ and $\{\omega_n, \omega_m, \omega_r\}$, where p_0, q_0, r_0 are independent energy variables (the arguments of the spectral functions) and $\omega_n, \omega_m, \omega_r$ are Matsubara frequencies constrained by the relation

$$\omega_n + \omega_m + \omega_r = 0. \tag{33}$$

There are three ways of parametrizing the solutions of this equation (by choosing 2 of the 3 internal Matsubara frequencies as the independent variables). Correspondingly, the fraction on the left hand side of Eq. (32) can be decomposed in the following sum of three simpler fractions, each term involving one choice of independent variables:

$$\begin{aligned}\frac{1}{(p_0 - i\omega_n)(q_0 - i\omega_m)(r_0 - i\omega_r)} &= \frac{1}{p_0 + q_0 + r_0} \\ &\times \left\{ \frac{1}{(p_0 - i\omega_n)(q_0 - i\omega_m)} + \frac{1}{(p_0 - i\omega_n)(q_0 - i\omega_r)} + \frac{1}{(p_0 - i\omega_m)(q_0 - i\omega_r)} \right\},\end{aligned} \tag{34}$$

the equality being valid when $\omega_n, \omega_m, \omega_r$ satisfy the relation (33). This decomposition (analogous to that in Eq. (27)) has a diagrammatic interpretation that we shall discuss more generally later.

This formula allows us to perform the sum over the Matsubara frequencies in Eq. (32) by summing in each term of the right hand side of (34) over the appropriate independent variables. As was the case in the previous example (see Eq. (27)), while the sum on the left hand side is well defined (since each independent frequency occurs there at least twice; see e.g. (29)), this is not so in the right hand side. As we did in Eq. (24), we then introduce a regulator in the form of exponential factors attached to each line, and write:

$$\begin{aligned}\frac{e^{i\omega_n\tau_n} e^{i\omega_m\tau_m} e^{i\omega_r\tau_r}}{(p_0 - i\omega_n)(q_0 - i\omega_m)(r_0 - i\omega_r)} &= \frac{1}{p_0 + q_0 + r_0} \\ \times \left\{ \frac{e^{i\omega_n(\tau_n-\tau_r)} e^{i\omega_m(\tau_m-\tau_r)}}{(p_0 - i\omega_n)(q_0 - i\omega_m)} \right. &+ \left. \frac{e^{i\omega_n(\tau_n-\tau_m)} e^{i\omega_r(\tau_r-\tau_m)}}{(p_0 - i\omega_n)(q_0 - i\omega_r)} + \frac{e^{i\omega_m(\tau_m-\tau_n)} e^{i\omega_r(\tau_r-\tau_n)}}{(p_0 - i\omega_m)(q_0 - i\omega_r)} \right\},\end{aligned} \tag{35}$$

where τ_n, τ_m, τ_r are arbitrary times. In each term of the right hand side of Eq. (34), we have used Eq. (33) to express the exponential factors in terms of the relevant independent Matsubara frequencies. As long as the various combinations of time do not vanish, the sums over Matsubara frequencies are now well defined. Once they are performed, we may take the limit $\tau \to 0$. The left hand side is well defined when $\tau \to 0$. In the right hand side, each

term has a limit that depends on the way we take the limit $\tau \to 0$ (see Eq. (6)). Of course, the sum of these 3 terms is independent of the way we take the limit. Let us consider for example the limit:

$$\tau_n = 3\theta \qquad \tau_m = 2\theta \qquad \tau_r = \theta \qquad \theta \,\to\, 0^+\,. \tag{36}$$

We obtain then:

$$\frac{1}{\beta^2}\sum_{\{n,m,r\}}\frac{1}{(p_0-i\omega_n)(q_0-i\omega_m)(r_0-i\omega_r)} = \frac{n_{p_0}n_{q_0}-n_{p_0}n_{-r_0}+n_{-q_0}n_{-r_0}}{r_0+p_0+q_0}, \tag{37}$$

where the notation $\sum_{\{n,m,r\}}$ is meant to indicate that the summation over the Matsubara frequencies is constrained by Eq. (33). Eq. (37) is identical to Eq. (32). Other choices of the limit would lead to distinct but equivalent expressions. This non uniqueness is of the same nature as that discussed after Eq. (22).

We now return to the sum-integral (29) and use Eq. (32) to write:

$$I = \int_p\int_q\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\frac{dq_0}{2\pi}\frac{dr_0}{2\pi}\rho(p_0,\mathbf{p})\rho(q_0,\mathbf{q})\rho(r_0,\mathbf{r})\frac{n_{p_0}n_{q_0}-n_{p_0}n_{-r_0}+n_{-q_0}n_{-r_0}}{r_0+p_0+q_0}. \tag{38}$$

Note that the expression (38) is well defined even in cases where the denominator vanishes, i.e., when $r_0+p_0+q_0=0$. This is because the numerator also vanishes (linearly) in this case, as is evident from Eq. (30). However, we shall find useful to be able to treat separately the various terms occuring in the numerator. In order to manipulate well defined quantities, it is necessary to introduce a regularisation, such as for instance a principal value prescription, or adding a small imaginary part to the denominator, i.e., replacing in Eq. (32) $1/(r_0+p_0+q_0)$ by $1/(r_0+p_0+q_0+i\alpha)$, where α is infinitesimal. Of course, it is only the sum of the three terms in Eq. (32) that is independent of α: individual contributions will contain imaginary parts depending on α, and will be different if we use instead a principal value prescription. We adopt in the following the regularisation which consists in adding a small imaginary part to the denominator. Then we can use the expression (3) of the propagator in terms of the spectral function in order to perform some energy integrations, and rewrite Eq. (38) as the following sum of three terms:

$$\begin{aligned} I \;=\; \int_p\int_q\Big\{&\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\frac{dq_0}{2\pi}\rho(p_0,\mathbf{p})\rho(q_0,\mathbf{q})n_{p_0}n_{q_0}D(-p_0-q_0-i\alpha,\mathbf{r})\\ &+\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\frac{dr_0}{2\pi}\rho(p_0,\mathbf{p})\rho(r_0,\mathbf{r})(-n_{p_0})n_{-r_0}D(-p_0-r_0-i\alpha,\mathbf{r})\\ &+\int_{-\infty}^{\infty}\frac{dq_0}{2\pi}\frac{dr_0}{2\pi}\rho(q_0,\mathbf{q})\rho(r_0,\mathbf{r})n_{-q_0}n_{-r_0}D(-q_0-r_0-i\alpha,\mathbf{r})\Big\}\,. \end{aligned} \tag{39}$$

This expression can be written directly by using the rules derived in the next section.

We turn now to our main goal which is to isolate the vacuum contributions. To do so, we express I in terms of statistical factors with positive arguments. To this aim, we start from Eq. (38), split each statistical factor in two pieces according to Eq. (10), and gather terms containing respectively zero, one and two statistical factors, that we denote respectively by $I^{(0)}$, $I^{(1)}$, and $I^{(2)}$. We get:

$$\begin{aligned} I \; = \; & \int_p \int_q \int_{-\infty}^{\infty} \frac{dp_0}{2\pi}\frac{dq_0}{2\pi}\frac{dr_0}{2\pi}\rho(p_0,\mathbf{p})\rho(q_0,\mathbf{p})\rho(r_0,\mathbf{r})\frac{1}{r_0+p_0+q_0+i\alpha} \\ & \qquad \times \{\theta(-p_0)\theta(-q_0) - \theta(-p_0)\theta(r_0) + \theta(q_0)\theta(r_0) \\ & \qquad +3\varepsilon(p_0)n_{|p_0|}\left[-\theta(-q_0)+\theta(r_0)\right] + 3\varepsilon(p_0)n_{|p_0|}\varepsilon(q_0)n_{|q_0|}\big\} \, . \end{aligned} \qquad (40)$$

The term which contains no thermal factors is the vacuum contribution $I^{(0)}$ to I. (The terminology refers here to the explicit temperature dependence; if the spectral density ρ is not the free spectral density ρ_0, it may depend on the temperature, generating implicit temperature dependence in $I^{(0)}$.) Note that there is no singularity in this term, the combination of θ functions in the numerator vanishing when $r_0+p_0+q_0=0$, as one can easily verify. Thus the $i\alpha$ may be omitted. In fact, this vacuum contribution may also be written as an Euclidean integral, which may be more convenient for its explicit calculation (at least in the case where $D=D_0$):

$$I^{(0)} = \int \frac{d^4p}{(2\pi)^4}\int \frac{d^4q}{(2\pi)^4} D(ip_0,\mathbf{p})D(iq_0,\mathbf{q})D(-ip_0-iq_0,\mathbf{k}-\mathbf{p}-\mathbf{q}). \qquad (41)$$

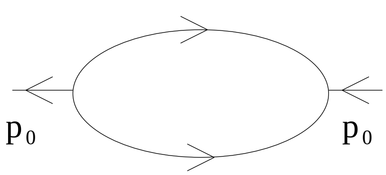


FIG. 4: A one-loop subdiagram of the two-loop diagram of Fig. 3

In the thermal contributions, singularities may arise in individual contributions and the $i\alpha$ must be kept in the denominator. The term with one thermal factor contains a vacuum one-loop contribution which is represented in Fig. 4. More precisely, we write:

$$\int_q \int_{-\infty}^{\infty} \frac{dq_0}{2\pi}\frac{dr_0}{2\pi}\rho(q_0,\mathbf{q})\rho(r_0,\mathbf{r})\,\frac{-\theta(-q_0)+\theta(r_0)}{p_0+q_0+r_0+i\alpha} = J_0(-p_0-i\alpha,\mathbf{p}), \qquad (42)$$

where J_0 is the one-loop integral given by Eq. (24) above (after proper analytic continuation). Then the contribution with one thermal factor is of the form:

$$I^{(1)} = 3\int \frac{d^3p}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\sigma(p_0,\mathbf{p})J_0(-p_0-i\alpha,\mathbf{p}). \tag{43}$$

The term with two thermal factors is given by:

$$I^{(2)} = 3\int_p\int_q\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\frac{dq_0}{2\pi}\sigma(p_0,\mathbf{p})\sigma(q_0,\mathbf{p})D(-p_0-q_0-i\alpha,\mathbf{r}), \tag{44}$$

and one can verify that the dependence on α disappears in the sum $I^{(1)}+I^{(2)}$, as it should.

Summarizing, one can write the integral I in Eq. (40) as $I = I^{(0)} + I^{(1)}(\alpha) + I^{(2)}(\alpha)$, where the two temperature dependent terms depend explicitly on the regulator α, while their sum does not. We should note also that we have been able to write the subdiagrams involved in the calculation of $I^{(1)}$ and $I^{(2)}$ as simple analytical continuation of vacuum n-point functions, $J_0(-p_0-i\alpha,\mathbf{p})$ and $D(-p_0-q_0-i\alpha,\mathbf{r})$ respectively. In the rest of this paper, we shall examine under which conditions such a strategy can be generalized. But before we do that, it is useful to recall the generalization of the method that we have used to calculate the sums over the Matsubara frequencies.

III. GENERAL RULES

The rules that we are about to describe have been derived long ago by M. Gaudin [4], but his work seems to have been largely ignored in the recent literature. We therefore find it appropriate to recall here the main steps involved in their derivation, without however going into all the subtleties of the complete proof which can be found in [4]. As we proceed, in order to make the discussion more concrete, we shall carry along a specific non trivial example, that of the two-loop diagram of Fig. 5.

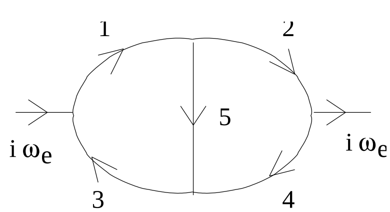


FIG. 5: A 2-loop contribution to the self-energy in ϕ^3 scalar field theory

Consider a general diagram Γ with N_I internal lines, N_V vertices and N_L loops. In order to avoid complications with multiple poles, we assume Γ to be two-particle irreducible, i.e., one cannot isolate self-energy insertions on the internal lines of Γ. We assume that each line of the diagram has been oriented and labelled (see the example in Fig. 5). The external frequencies will be denoted collectively by $\{\omega_e\}$, and the internal frequencies by $\{\omega_i\}$ with $i = 1, \cdots, N_I$. The evaluation of the diagram involves that of the following sum over the Matsubara frequencies:

$$I\{i\omega_e\} = \frac{1}{\beta^{N_L}} \sum_{\{n_i\}} \prod_{i=1}^{N_I} D(i\omega_i), \tag{45}$$

where the notation $\sum_{\{n_j\}}$ stands for a sum restricted to N_L independent Matsubara frequencies (we leave aside the momentum integrals which play no role in the present discussion). The choice of independent variables is at this point left unspecified. The N_I internal Matsubara frequencies ω_i obey $N_V - 1$ independent linear relations (also involving $\{i\omega_e\}$):

$$R_v\{i\omega_e, i\omega_i\} = 0, \text{ for } v = 1, \ldots, N_V - 1, \tag{46}$$

which reflect the conservation of energies at each vertex. In R_v, a frequency ω appears with a positive sign if it is attached to a line which enters the vertex v, with a negative sign if the corresponding line leaves the vertex. The number of independent variables is $N_I - N_V + 1 = N_L$.

We then express each propagator in terms of its spectral function (see Eq. (3)), and write:

$$I\{i\omega_e\} = \frac{1}{\beta^{N_L}} \sum_{\{n_i\}} \prod_{i=1}^{N_I} \int \frac{dp_i^0}{2\pi} \rho(p_i^0) \frac{1}{p_i^0 - i\omega_i}. \tag{47}$$

At this point, to each internal line i of the diagram are attached two energy variables: a real variable p_i^0, argument of a spectral function, and a Matsubara frequency ω_i. While the real variables p_i^0 are independent, this is not so for the Matsubara frequencies which satisfy Eqs. (46). The problem is then to compute the following sum:

$$\frac{1}{\beta^{N_L}} \sum_{\{n_i\}} \prod_{i=1}^{N_I} \frac{1}{p_i^0 - i\omega_i}, \tag{48}$$

where the sum $\sum_{\{n_i\}}$ is restricted to Matsubara frequencies satisfying Eqs. (46).

We proceed by generalizing Eq. (34), and recall how the system of linear equations (46) can be solved by exploiting the notion of tree diagrams [4]. Given a connected diagram Γ, a

tree is a set of lines of Γ joining all the vertices and making a connected graph without loops. It can be shown that each set of independent variables that can be chosen in order to express the solutions of Eqs. (46) can be associated with one of the trees that can be identified on the diagram considered. We denote by $\mathcal{T}$ the set of lines which belong to a given tree and by $\bar{\mathcal{T}}$ the set of lines which do not belong to $\mathcal{T}$. There are $N_V - 1$ lines in $\mathcal{T}$ and N_L lines in $\bar{\mathcal{T}}$. For a given tree $\mathcal{T}$, the N_L independent variables are the Matsubara frequencies attached to the lines of $\bar{\mathcal{T}}$; we shall denote them by $\{\omega_l\}$. The remaining variables $\{\omega_j\}$, with $j \in \mathcal{T}$, are linear combinations Ω_j of the independent internal Matsubara frequencies $\{\omega_l\}$ and the external Matsubara frequencies $\{\omega_e\}$:

$$j \in \mathcal{T}, \;\; l \in \bar{\mathcal{T}}, \qquad \omega_j = \Omega_j \left\{ i\omega_e, i\omega_l \right\}. \tag{49}$$

There is a simple way to read the values of the frequencies $\Omega_j\{i\omega_e, i\omega_l\}$ on a graph: $i\Omega_j$ is the algebraic sum of all the energies ($i\omega_l$ for the lines of $\bar{\mathcal{T}}$ and $i\omega_e$ for the external lines) which flow through the (oriented) branch j of $\mathcal{T}$.

We then have the following formula that allows us to reduce the rational fraction of Eq. (48), when the ω_i's satisfy Eqs. (46):

$$\prod_{i=1}^{N_I} \frac{1}{p_i^0 - i\omega_i} = \sum_{\mathcal{T}} \prod_{j\in T} \frac{1}{p_j^0 - i\Omega_j \left\{ i\omega_e, p_l^0 \right\}} \prod_{l\in\bar{\mathcal{T}}} \frac{1}{p_l^0 - i\omega_l}, \tag{50}$$

where the sum over the trees corresponds to the sum over all possible sets of independent internal Matsubara frequencies. In this formula, the entries $i\omega_l$ in $i\Omega_j\{i\omega_e, i\omega_l\}$ of Eq. (49) have been replaced by the corresponding real energies p_l^0.

As an illustration we show in Fig. 6 the various trees corresponding to the 2-loop diagram in Fig. 5. With the labelling of Fig. 5, the independent frequencies for the first tree in Fig. 6 are ω_1 and ω_2 and we have: $i\Omega_3 = p_1^0 - i\omega_e$, $i\Omega_4 = p_2^0 - i\omega_e$, $i\Omega_5 = p_1^0 - p_2^0$. Thus the formula (50) for the first tree yields:

$$\prod_{i=1}^{5} \frac{1}{p_i^0 - i\omega_i} \longrightarrow \frac{1}{p_3^0 - p_1^0 + i\omega_e} \, \frac{1}{p_4^0 - p_2^0 + i\omega_e} \, \frac{1}{p_5^0 - p_1^0 + p_2^0} \, \frac{1}{p_1^0 - i\omega_1} \, \frac{1}{p_2^0 - i\omega_2}. \tag{51}$$

At this point, the sum over the Matsubara frequencies, Eq.(48), reduces, for each tree, to:

$$\frac{1}{\beta^{N_L}} \prod_{l=1}^{N_L} \sum_{n_l} \frac{1}{p_l^0 - i\omega_l}, \tag{52}$$

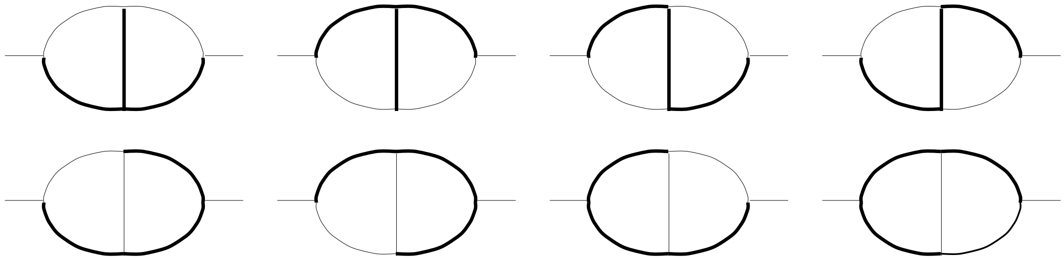


FIG. 6: The various trees of the two-loop diagram of Fig. 5 are represented by thick lines. The thin lines are those of $\bar{\mathcal{T}}$: they carry the independent Matsubara frequencies.

where now the ω_l's are independent variables, and we have set $\omega_l = 2\pi n_l T$. These sums are ill-defined in the absence of a regulator. We proceed as in the examples of the previous section and attach to each internal line a time τ_i that we shall let go to zero at the end of the calculation, transforming Eq. (50) into:

$$\prod_{i=1}^{N_I} \frac{\mathrm{e}^{i\omega_i\tau_i}}{p_i^0 - i\omega_i} = \sum_{\mathcal{T}} \left(\mathrm{e}^{i\omega_e T_e} \prod_{j\in\mathcal{T}} \frac{1}{p_j^0 - i\Omega_j\left\{i\omega_e, p_l^0\right\}} \prod_{l\in\bar{\mathcal{T}}} \frac{\mathrm{e}^{i\omega_l T_l}}{p_l^0 - i\omega_l} \right), \tag{53}$$

where we have set:

$$\sum_i \omega_i\tau_i = \sum_{l\in\bar{\mathcal{T}}} \omega_l T_l + \omega_e T_e. \tag{54}$$

Both T_e and T_l are linear combinations of the times τ_i. We shall need in fact only T_l, and this is determined by the following simple rule.

First we observe that each line l of $\bar{\mathcal{T}}$ defines a unique loop of Γ. To determine the linear combination T_l we consider successively all the lines of the loop l: a line k in this loop contributes $+\tau_k$ if it is oriented as the line l and $-\tau_k$ in the opposite case. Thus, in the example discussed above, there are two loops. The phase factor reads $\sum_i \omega_i\tau_i = \omega_1 T_1 + \omega_2 T_2 + \omega_e T_e$ with $T_1 = \tau_1 + \tau_3 + \tau_5$, $T_2 = \tau_2 + \tau_4 - \tau_5$ and $T_e = -\tau_3 - \tau_4$ (to obtain this result we have expressed ω_3, ω_4 and ω_5 in terms of ω_1, ω_2 and ω_e, the independent variables corresponding to the first tree in Fig. 6).

We are now ready to perform the sum over independent Matsubara frequencies in Eq. (53). The left hand side is well defined, even in the absence of a regulator, so that the limit $\tau_i \to 0$ can be taken, and the result is independent of the ways the various τ_i's approach 0. In the right hand side, the regulator matters, and the results of individual sums depend on the sign

ϵ_l of T_l (see Eq. (6)):

$$\frac{1}{\beta}\sum_{n_l}\frac{e^{i\omega_l T_l}}{p_l^0 - i\omega_l} = \epsilon_l n_{\epsilon_l p_l^0}\, e^{p_l^0 T_l}. \tag{55}$$

The factor $e^{i\omega_e T_e}$ has no influence in the limit $\tau_i \to 0$ and we can forget it; the same remark applies to the factor $e^{p_l^0 T_l}$ in the right hand side. Thus, the sum-integral (47) can be written as follows:

$$I\left\{i\omega_e\right\} = \int \prod_{i=1}^{N_I} \frac{dp_i^0}{(2\pi)}\rho(p_i^0)\sum_{\mathcal{T}}\left(\prod_{j\in\mathcal{T}}\frac{1}{p_j^0 - i\Omega_j}\prod_{l\in\bar{\mathcal{T}}}\epsilon_l n_{\epsilon_l p_l^0}\right). \tag{56}$$

This formula is essentially that derived by Gaudin [4]. It can be translated into a set of rules listed below.

Rules

1. Determine the family of all the trees $\mathcal{T}$ that can be drawn on Γ. A tree is a connected set of lines of Γ which joins all the vertices and contains no loop. A tree contains $N_V - 1$ lines, with N_V the number of vertices of Γ. We call $\bar{\mathcal{T}}$ the set of lines of Γ which do not belong to $\mathcal{T}$. The external lines play no role in the determination of the trees: they belong neither to $\mathcal{T}$, nor to $\bar{\mathcal{T}}$.

2. Specify the orientation of each line of Γ, and affect to each line k, and once and for all, a positive number τ_k. We define the orientation of a loop by the following rule. Given a tree $\mathcal{T}$, consider the loop l associated to the line l of $\bar{\mathcal{T}}$. The orientation of the loop is the algebraic sum of the τ_k carried by each line of the loop, with τ_k counted positively if the line k is oriented as the line l, and negatively (as $-\tau_k$) in the opposite case. The choice of the τ_k must be such that the orientation of each loop that can be drawn on Γ is non vanishing (it is always possible to do so). Note that loops and their orientations may change depending on the tree one begins with.

3. The contribution of the sum over Matsubara frequencies to Γ is then given by the formula (56). It is the sum of the contributions of the various trees $\mathcal{T}$. To each line l of $\bar{\mathcal{T}}$ is associated an integral over the energy p_l^0 with the weight $\rho(p_l^0)\epsilon_l n_{\epsilon_l p_l^0}$ where $\epsilon_l = +1$ or -1 depending on whether the orientation of the loop l is, respectively, positive or negative. To each line j of $\mathcal{T}$ is associated a factor $\rho(p_j^0)\,(p_j^0 - i\Omega_j)^{-1}$ where Ω_j is determined as follows: given the orientation of the line j, $i\Omega_j$ is the algebraic sum of the quantities $i\omega_e$ and p_l^0, carried respectively by the external lines and the

lines of $\bar{\mathcal{T}}$, which flow through the line j in the direction specified on the line j. Note that various choices of τ_k may lead to seemingly different, but equivalent, expressions (since the numerators depend on the orientation of the loops, and hence on the specific choice of the τ_k's). We emphasize that the ambiguity concerns only the signs in front of the statistical factors and their arguments. In particular, it is important to observe that, in all cases, each statistical factor carries a single frequency attached to a line (and not sums of such frequencies).

These rules apply indifferently at finite temperature and at zero temperature. In the latter case, the factor to be associated to each loop integral is $-\rho(p_l^0)\epsilon_l\theta(-\epsilon_l p_l^0)$ (rather than $\rho(p_l^0)\epsilon_l n_{\epsilon_l p_l^0}$).

One can verify that these rules are satisfied on the examples discussed in the previous section (see Eqs. (22) and (38)). To apply them to our two-loop example, we first define a set of regulators τ_k; in the present case the choice $\tau_k = k\theta^+$ (with k an integer and $\theta \to 0$) is a possible one. Then we can compute for instance the numerator for the first tree in Fig. 6. Consider the loop involving the line 1, i.e., the set of lines $\{1, 5, 3\}$; its orientation, $(1+5+3)\theta$, is positive, so that the corresponding contribution to the numerator is $n_{p_1^0}$. The orientation of the loop $\{2, 4, 5\}$ is also positive ($(2+4-5)\theta$), resulting in the contribution $n_{p_2^0}$ to the numerator. Combining with the denominator obtained from Eq. (51), we obtain the contribution of the first tree as:

$$\frac{n_{p_1^0} n_{p_2^0}}{(p_3^0 - p_1^0 + i\omega_e)(p_5^0 + p_2^0 - p_1^0)(p_4^0 - p_2^0 + i\omega_e)}. \tag{57}$$

Repeating this simple procedure for all the trees in Fig. 6 we obtain:

$$
\begin{aligned}
I(i\omega_e) = \int_{12345} & \left\{ \frac{n_1 n_2}{(p_3^0 - p_1^0 + i\omega_e)(p_5^0 + p_2^0 - p_1^0)(p_4^0 - p_2^0 + i\omega_e)} \right. \\
& + \frac{n_3 n_4}{(p_1^0 - p_3^0 - i\omega_e)(p_5^0 + p_4^0 - p_3^0)(p_2^0 - p_4^0 - i\omega_e)} \\
& + \frac{n_3 n_2}{(p_1^0 - p_3^0 - i\omega_e)(p_5^0 + p_2^0 - p_3^0 - i\omega_e)(p_4^0 - p_2^0 + i\omega_e)} \\
& + \frac{n_1 n_4}{(p_3^0 - p_1^0 + i\omega_e)(p_5^0 - p_1^0 + p_4^0 + i\omega_e)(p_2^0 - p_4^0 - i\omega_e)} \\
& + \frac{n_1 n_{-5}}{(p_3^0 - p_1^0 + i\omega_e)(p_2^0 + p_5^0 - p_1^0)(p_4^0 + p_5^0 - p_1^0 + i\omega_e)} \\
& + \frac{n_3 n_{-5}}{(p_1^0 - p_3^0 - i\omega_e)(p_4^0 + p_5^0 - p_3^0)(p_2^0 - p_3^0 + p_5^0 - i\omega_e)} \\
& + \frac{n_2 n_5}{(p_3^0 - p_2^0 - p_5^0 + i\omega_e)(p_1^0 - p_2^0 - p_5^0)(p_4^0 - p_2^0 + i\omega_e)} \\
& \left. + \frac{n_4 n_5}{(p_1^0 - p_4^0 - p_5^0 - i\omega_e)(p_3^0 - p_4^0 - p_5^0)(p_2^0 - p_4^0 - i\omega_e)} \right\},
\end{aligned}
\tag{58}
$$

with the short-hand notations $n_i = n_{p_i^0}$ and $n_{-i} = -n_{-p_i^0}$ and

$$
\int_{12345} \equiv \int \prod_{i=1}^{5} \frac{dp_i^0}{2\pi} \rho(p_i^0). \tag{59}
$$

IV. EXPANSION IN THE NUMBER OF STATISTICAL FACTORS

We now return to our main goal, which is to isolate vacuum contributions in a general Feynman diagram at finite temperature. The rules of the previous section enable us to do that easily. They also allow us to see that it is not always possible to identify the vaccum subdiagrams with analytic continuation of simple vacuum amplitudes.

To proceed with the separation of explicit temperature dependent contributions, we use Eq. (10) to isolate the temperature dependent factor in $\epsilon_l n_{\epsilon_l p_l^0}$:

$$
\epsilon_l n_{\epsilon_l p_l^0} = -\epsilon_l \theta(-\epsilon_l p^0) + \epsilon(p^0) n_{|p^0|}. \tag{60}
$$

Next, in the contribution of each tree in Eq. (56), we replace the quantity $\epsilon_l n_{\epsilon_l p^0}$ attached to each line l of $\bar{\mathcal{T}}$ by its expression above, and separate the various terms thus obtained. One gets then, for each tree, 2^{N_L} contributions containing terms with $0, \ldots, N_L$ factors $\epsilon(p_l^0) n_{|p_l^0|}$. Diagrammatically the operation is illustrated in Fig. 7 for our two-loop example: the total

number of contributions is $2^2 \times 8 = 32$, each tree generating one vacuum contribution, two contributions with one statistical factor, and one contribution with two statistical factors. The lines of $\bar{\mathcal{T}}$ carrying vacuum factors $-\epsilon_l \theta(-\epsilon_l p^0)$ are represented by thin lines; we shall call them "vacuum lines". Those carrying a statistical factor $\epsilon(p_l^0) n_{|p_l^0|}$ are represented by dotted lines; we shall call them "thermal lines". Note that in each column in Fig. 7, a given tree occurs once and only once.

In order to proceed further, we need to analyze the contributions of subsets of terms, for instance those which contain a given thermal line. In other words, as suggested by the way the various diagrams are grouped in Fig. 7, we wish to give a meaning to sums of terms which involve only a subset of the trees of the diagram, that is, we would like to manipulate independently all the trees. This raises problems that we now discuss.

A. Regularized summation over the tree diagrams

Returning to the formula (56), we note that some denominators vanish for some particular values of the variables p_j^0, leading to potential singularities. However such singularities are fictitious: indeed, the sum of all tree contributions is well defined even when denominators vanish (for an example, see Sect. II C). That there are, in the original diagram, no singularities associated with vanishing denominators is clear in the time representation: a vanishing denominator would correspond in fact to an integral proportional to β, leading to no denominator! (We should not confuse such fictitious singularities with those which may occur for certain real values of the external frequencies and which are associated with physical processes). Now, since denominators can vanish in individual trees for some values of the integration variables p_i^0, it is not possible to do the integration over the p_i^0 before doing the sum over trees. If we wish to do so and be able to manipulate independently the contributions of the various trees, we need to introduce a regularization. We have met this problem after Eq. (38), and we shall proceed similarly in the general case by attaching a small imaginary part to the various denominators. There is some arbitrariness in doing that, the only constraint being that the final result (i.e. including the sum over all trees) should be independent of the choice of regulators. This constraint implies in particular that a given denominator occuring in different trees must carry everywhere the same imaginary part. One way to guarantee this is to add a small imaginary part $i\alpha_j$ to all the variables

p_j^0 of the internal lines [8]. Note that the regulators thus introduced may not be all needed (nor chosen independently). For instance, in the expression (58) for the two-loop example, there are only two "dangerous" denominators, namely $(p_5^0+p_2^0-p_1^0)$ and $(p_5^0+p_4^0-p_3^0)$. The other denominators contain the imaginary frequency $i\omega_e$ and cannot vanish. Thus we need a priori only two regulators, namely the combinations $\alpha_5+\alpha_2-\alpha_1$ and $\alpha_5+\alpha_4-\alpha_3$. But these particular combinations should not vanish, which places a constraint on the choice of the α_j.

Assuming such a regularization, we can perform trivially, in Eq. (56), the N_I-N_L integrals over the spectral densities attached to the lines of a given tree so as to reconstruct a propagator D for each line of the tree. The final result reads:

$$I\{i\omega_e\}=\sum_{\mathcal{T}}\int\prod_{l\in\bar{\mathcal{T}}}\frac{dp_l^0}{(2\pi)}\rho(p_l^0)\epsilon_l n_{\epsilon_l p_l^0}\prod_{j\in\mathcal{T}}D(i\Omega_j;\alpha), \tag{61}$$

where α denotes collectively the set of regulators.

At this point we return to the general discussion and note that a given tree generates $C_{N_L}^{N_l}$ terms with N_l factors $\sigma(p_l^0)\equiv\rho(p_l^0)\epsilon(p_l^0)n_{|p_l^0|}$ $(0\le N_l\le N_L)$. For a given N_l we consider all the possible sets $\mathcal{A}$ of thermal lines (carrying σ factors and which we label by the variables p_a^0). The complementary sets $\bar{\mathcal{A}}$ are (N_I-N_l)-loop connected subdiagrams (in general one line-reducible). Examples are given in Fig. 7 (for instance, the various sets $\mathcal{A}$ with one statistical factor are the five blocks labelled $1,2,\cdots,5$ in the middle columns of Fig. 7). We then write:

$$\begin{aligned}I\{i\omega_e\} &= \sum_{N_l=0}^{N_L}\sum_{\mathcal{A}_{N_l}}\int\prod_{a\in\mathcal{A}_{N_l}}\frac{dp_a^0}{2\pi}\sigma(p_a^0)I_{\mathcal{A}_{N_l}}\left\{i\omega_e,p_a^0;\alpha\right\}\\ &= \sum_{N_l=0}^{N_L}I^{(N_l)}\{\omega_e;\alpha\}\,,\end{aligned} \tag{62}$$

where $I_{\mathcal{A}_{N_l}}\{i\omega_e,p_a^0;\alpha\}$ is the sum of all the contributions to $I\{i\omega_e\}$ which contain the same set $\mathcal{A}_{N_l}$ of N_l thermal lines, while $I^{(N_l)}\{\omega_e;\alpha\}$ is the sum of all the contributions containing N_l statistical factors. In our two-loop example, the sets $\mathcal{A}$ contain 0,1 or 2 thermal lines, and we can write, more explicitly:

$$I\{i\omega_e\}\;=\;I^{(0)}\{i\omega_e;\alpha\}+I^{(1)}\{i\omega_e;\alpha\}+I^{(2)}\{i\omega_e;\alpha\}\,, \tag{63}$$

with

$$I^{(1)}\{i\omega_e;\alpha\}=\sum_a\int\frac{dp_a^0}{2\pi}\sigma(p_a^0)I_a\left\{i\omega_e,p_a^0;\alpha\right\}, \tag{64}$$

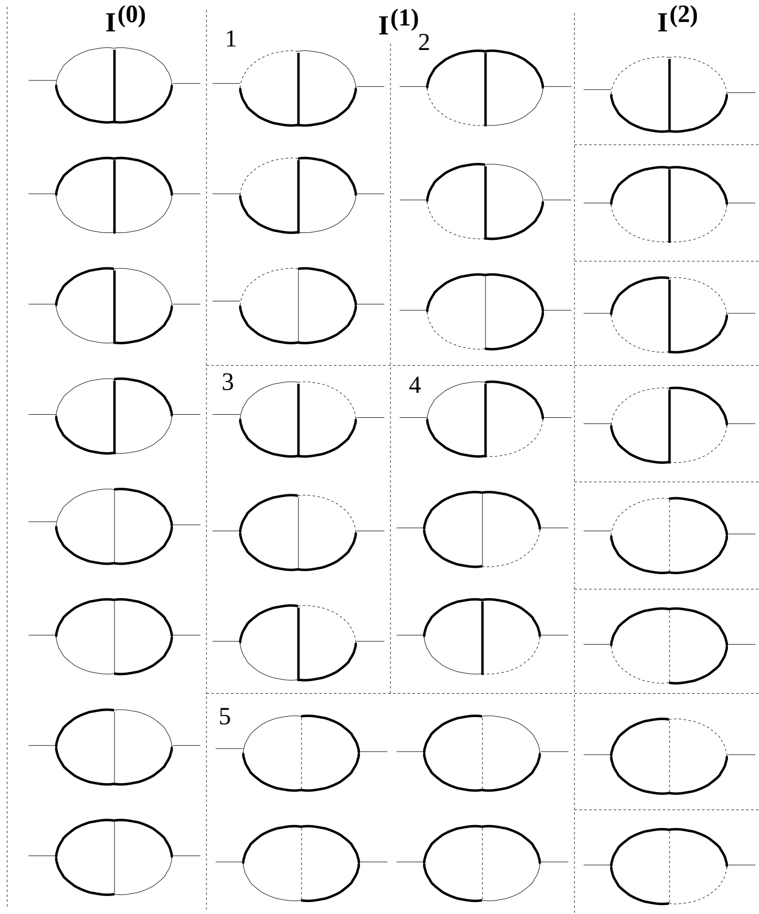


FIG. 7: The contributions to the 2-loop diagram of Fig. 5 with zero ($I^{(0)}$), one ($I^{(1)}$) and two ($I^{(2)}$) thermal factors. The thick lines represent the trees. The thin lines are associated with zero temperature numerators, the dotted lines with thermal factors. The first column contains all the zero temperature contributions. The last column contains the contributions with two thermal factors. The columns in the middle contains the contributions with one thermal factor, grouped by blocks where the thermal factor is attached to a given line of the diagram.

and

$$I^{(2)}\left\{i\omega_e;\alpha\right\} = \sum_{(a,b)}\int \frac{dp_a^0}{2\pi}\frac{dp_b^0}{2\pi}\sigma(p_a^0)\sigma(p_b^0)I_{ab}\left\{i\omega_e,p_a^0,p_b^0;\alpha\right\}, \tag{65}$$

where the subscripts $a, b, \cdots$ label the selected thermal lines. This formula has a simple interpretation. The terms with one statistical factor are obtained by replacing successively each internal line by a thermal line: the five ways to do this correspond to the five blocks in the middle columns of Fig. 7. Similarly for the terms with two statistical factors: the sum is

over all pairs (a, b) such that the remaining lines constitute a tree. Clearly these contributions are in one-to-one correspondence with the various trees. The dependence on α in I_a and I_{ab} reminds us that the separation of contributions with different numbers of thermal factors is well defined except for specific values of the energies p_a^0 for which denominators vanish: at these points combinations of statistical factors vanish, destroying the classification of the contributions according to the number of statistical factors they contain.

We will associate to a given $\mathcal{A}_{N_l}$ a n-point function (with $n = 2N_l + N_e$, where N_e is the number of external lines) $J_{\mathcal{A}_{N_l}} \{i\omega_e, i\omega_a, i\omega_a'\}$ corresponding to the computation of the diagram $\bar{\mathcal{A}}_{N_l}$ in the imaginary time formalism, at zero temperature. This diagram is obtained by cutting the lines of $\mathcal{A}_{N_l}$ and attributing to the two ends of the cut line, carrying initially the real variable p_a^0, two independent complex variables $i\omega_a$ and $i\omega_a'$. The question to be addressed in this section is whether the regularized integral $I_{\mathcal{A}_{N_l}} \{i\omega_e, p_a^0; \alpha\}$ can be considered as an analytic continuation of $J_{\mathcal{A}_{N_l}} \{i\omega_e, i\omega_a, i\omega_a'\}$ for suitably chosen variables $i\omega_a$ and $i\omega_a'$.

Let us first verify that $I_{A_{N_l}}$ and $J_{A_{N_l}}$ are given by (almost) identical rules. Consider a particular set $\mathcal{A}$ with N_l thermal lines. There exists a tree $\mathcal{T}$ on Γ such that the thermal lines belong to $\bar{\mathcal{T}}$, which also contains $N_L - N_l$ vacuum lines to which are associated factors $-\epsilon_j \theta(-\epsilon_j p_j^0)$. Consider the set $\mathcal{T}_{\mathcal{A}}$ of all the trees $\mathcal{T}'$ whose complements contain the same thermal lines. Clearly $\mathcal{T}_{\mathcal{A}}$ contains all the trees contributing to the $n-$point function $J_{\mathcal{A}} \{i\omega_e, i\omega_a, i\omega_a'\}$. Thus the denominators of J_A have the same structure as those of I_A, they differ solely in that in $J_{\mathcal{A}} \{i\omega_e, i\omega_a, i\omega_a'\}$ we have attached independent complex variables on the external lines, while in $I_{\mathcal{A}} \{i\omega_e, p_a^0; \alpha\}$ there is a unique real frequency p_a^0 attached to both ends of the thermal line (to within the $i\alpha_a$ inherited form the regularization). As for the numerators they are identical, to within the usual ambiguity related to the choice of the loop orientations. This follows from the fact that the σ factors do not depend on the choice of the orientation: thus the sign is determined by the orientation of the vacuum lines only, and these are the lines of $\bar{\mathcal{A}}$.

To proceed now it is best to look at specific examples. We shall consider next our two-loop example for which we can carry through successfully our analysis, and show indeed that the separation of vacuum contributions allows for a very simple discussion of ultraviolet divergences. At the end of this section, we shall discuss a counter-example showing that it is not always possible to do so.

B. The two-loop example

Consider then the various contributions to the two-loop diagram, as displayed in Fig. 7 The first column in Fig. 7 lists the terms with no thermal line. Their sum I_0 is nothing but the zero temperature limit of the two-loop diagram. It can be calculated with the rule given above. Alternatively, it can be written as an Euclidean integral (i.e. not performing the frequency integral first, but doing the calculation with covariant techniques). One gets then:

$$I^{(0)}(K) = \int \frac{d^4P}{(2\pi)^4} \int \frac{d^4Q}{(2\pi)^4} D(P)D(P-K)D(P-Q)D(Q-K)D(Q), \tag{66}$$

where the notation is that of Eq. (25).

Consider next the sum of the contributions with one thermal line that are associated with the diagrams of the bloc 1 in Fig. 7). We write this as

$$\int \frac{dp_1^0}{2\pi} \sigma(p_1^0) I_1(i\omega_e, p_1^0; \alpha), \tag{67}$$

with

$$I_1(i\omega_e, p_1^0; \alpha) = D(p_1^0 + i\alpha_1 - i\omega_e) L(i\omega_e, p_1^0; \alpha). \tag{68}$$

A diagrammatic representation of $I_1(i\omega_e, p_1^0; \alpha)$ is given in Fig. 9 below. In I_1, we have isolated the common propagator $D(p_1^0 + i\alpha_1 - i\omega_e)$, and L is defined by

$$\begin{aligned} L(i\omega_e, p_1^0; \alpha) \equiv \int_{245} & \left\{ \frac{-\theta(-p_2^0)}{(p_5^0 + p_2^0 - p_1^0)(p_4^0 - p_2^0 + i\omega_e)} \right. \\ & + \frac{-\theta(-p_4^0)}{(p_5^0 - p_1^0 + p_4^0 + i\omega_e)(p_2^0 - p_4^0 - i\omega_e)} \\ & \left. + \frac{\theta(p_5^0)}{(p_2^0 + p_5^0 - p_1^0)(p_4^0 + p_5^0 - p_1^0 + i\omega_e)} \right\}. \end{aligned} \tag{69}$$

The notation $\int_{245}$ is that introduced in Eq. (59). In the denominators, it is understood that all the variables p_j^0 are shifted by a small imaginary part ($p_j^0 \to p_j^0 + i\alpha_j$), in agreement with the regularization introduced in the previous subsection.

At this point we consider a related diagram, that of the three-point function in Fig. 8, computed at zero temperature, in imaginary time (we use the convention of incoming external momenta):

$$\Lambda(P_a, P_b) = \int \frac{d^4Q}{(2\pi)^4} D(P_a - Q) D(Q + P_b) D(Q). \tag{70}$$

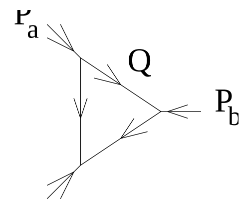


FIG. 8: One-loop contribution to the three-point function.

The notation for quadrimomenta is the same as in Eq. (25), that is, $P_a \equiv (i\omega_a, \mathbf{p}_a)$, $Q \equiv (iq_0, \mathbf{q})$. The integral in Eq. (70) can also be calculated by applying the rules of Sect. III at zero temperature. One gets (omiting the spatial momenta):

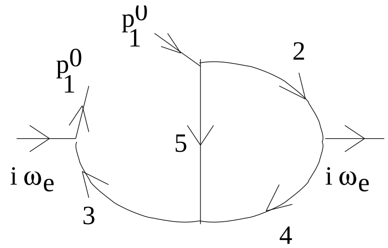
$$\Lambda(i\omega_a, i\omega_b) = \int_{245} \left\{ \frac{-\theta(-p_2^0)}{(p_5^0 + p_2^0 - i\omega_a)(p_4^0 - p_2^0 - i\omega_b)} + \frac{-\theta(-p_4^0)}{(p_5^0 + p_4^0 - i\omega_a - i\omega_b)(p_2^0 - p_4^0 + i\omega_b)} + \frac{\theta(p_5^0)}{(p_2^0 + p_5^0 - i\omega_a)(p_4^0 + p_5^0 - i\omega_a - i\omega_b)} \right\}. \tag{71}$$

Thus defined, $\Lambda(i\omega_a, i\omega_b)$ is an analytic function of the variables $i\omega_a, i\omega_b$, with singularities on the planes defined by $\mathrm{Im}(i\omega_a) = 0$, $\mathrm{Im}(i\omega_a) = 0$ or $\mathrm{Im}(i\omega_a + i\omega_b) = 0$. Alternatively, if one sets $i\omega_a = p_a^0 + i\alpha_a$, $i\omega_b = p_a^0 + i\alpha_b$, there are six domains of analyticity, depending on the relative signs of α_a, α_b and $\alpha_a + \alpha_b$. Now, it is not difficult to find a set of regulators making it possible to identify L with Λ in one of its domains of analyticity. By comparing Eqs. (71) and (69), one finds the relations:

$$i\omega_a = p_1^0 + i(\alpha_1 - \alpha_2 - \alpha_5) \qquad i\omega_b = -i\omega_e + i(\alpha_2 - \alpha_4). \tag{72}$$

By choosing $\alpha_2 - \alpha_4 = 0$, and $\alpha_2 + \alpha_5 = 0$, on may then write $L(i\omega_e, p_1^0; \alpha) = \Lambda(p_1^0 + i\alpha_1, -i\omega_e)$.

One can treat I_2, I_3 and I_4 in the same way (one obtains the further conditions $\alpha_4 + \alpha_5 = 0$, $\alpha_5 - \alpha_1 = 0$, $\alpha_5 - \alpha_3 = 0$, $\alpha_1 - \alpha_3 = 0$ which are compatible with the previous ones). For I_5, corresponding to the bottom diagram in Fig. 9, a similar analysis can be carried out. One



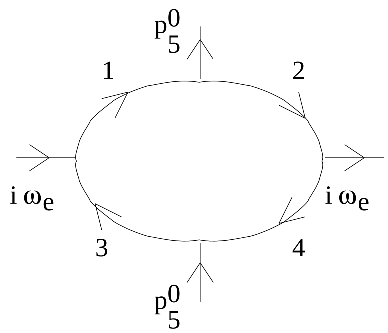


FIG. 9: Diagrammatic representations of the contributions to $I_1(i\omega_e, p_1^0; \alpha)$ (top, see Eq. (68)), and $I_5(i\omega_e, p_5^0; \alpha)$ (bottom, see Eq. (73)).

has:

$$I_5 \left\{i\omega_e, p_5^0; \alpha\right\} = \int_{1234} \left\{\frac{-\theta(-p_1^0)}{(p_3^0 - p_1^0 + i\omega_e)(p_2^0 + p_5^0 - p_1^0)(p_4^0 + p_5^0 - p_1^0 + i\omega_e)} + \frac{-\theta(-p_3^0)}{(p_1^0 - p_3^0 - i\omega_e)(p_4^0 + p_5^0 - p_3^0)(p_2^0 - p_3^0 + p_5^0 - i\omega_e)} + \frac{-\theta(-p_2^0)}{(p_3^0 - p_2^0 - p_5^0 + i\omega_e)(p_1^0 - p_2^0 - p_5^0)(p_4^0 - p_2^0 + i\omega_e)} + \frac{-\theta(-p_4^0)}{(p_1^0 - p_4^0 - p_5^0 - i\omega_e)(p_3^0 - p_4^0 - p_5^0)(p_2^0 - p_4^0 - i\omega_e)}\right\}, \tag{73}$$

where again it is understood that all the variables p_1, p_2, p_3, p_4 have a small imaginary part. One considers then the 4-point function associated to the bottom diagram in Fig. 9, calcu-

lated at zero temperature. By applying the rules of Sect. III one obtains

$$J_5\left\{i\omega_a, i\omega_b, i\omega_c\right\} = \int_{1234} \left\{ \frac{-\theta(-p_1^0)}{(p_3^0 - p_1^0 + i\omega_a)(p_2^0 - p_1^0 - i\omega_b)(p_4^0 - p_1^0 - i\omega_b - i\omega_c)} \right.$$
$$+ \frac{-\theta(-p_3^0)}{(p_1^0 - p_3^0 - i\omega_a)(p_4^0 - p_3^0 - i\omega_a - i\omega_b - i\omega_c)(p_2^0 - p_3^0 - i\omega_a - i\omega_b)}$$
$$+ \frac{-\theta(-p_2^0)}{(p_3^0 - p_2^0 + i\omega_a + i\omega_b)(p_1^0 - p_2^0 + i\omega_b)(p_4^0 - p_2^0 - i\omega_c)}$$
$$\left. + \frac{-\theta(-p_4^0)}{(p_1^0 - p_4^0 + i\omega_b + i\omega_c)(p_3^0 - p_4^0 + i\omega_a + i\omega_b + i\omega_c)(p_2^0 - p_4^0 + i\omega_c)} \right\}. \tag{74}$$

Again it is possible to find a set of regulators and a domain of analyticity of J_5 where J_5 and the regulated integral I_5 coincide. By comparing Eqs. (73) and (74), one finds the following relations:

$$i\omega_a = i\omega_e - i\alpha_1 + i\alpha_3 \qquad i\omega_b = -p_5^0 + i\alpha_1 - i\alpha_2 - i\alpha_5 \qquad i\omega_c = -i\omega_e + i\alpha_2 - i\alpha_4. \tag{75}$$

These allow us to write

$$I_5(i\omega_e, p_5^0; \alpha) = J_5(i\omega_e, -p_5^0 - i\alpha_5, -i\omega_e), \tag{76}$$

where the constraints discussed before Eq. (61) can be satisfied with the choice $\alpha_2 = \alpha_4$, $\alpha_1 = \alpha_3$, and $\alpha_5 \neq 0$ (note that $\alpha_2 = -\alpha_1$ in order for these conditions to be compatible with those derive for I_1, I_2, I_3 and I_4).

Finally, for completeness, we consider the terms with two factors σ (single blocs in Fig. 7), which raise in fact no real problems. Using the spectral representation for the propagator we perform all the integrals over frequencies which are not involved in σ. The resulting contribution takes the form of a product of propagators. For instance, I_{12} can be written as follows:

$$I_{12}(i\omega_e, p_1^0, p_2^0; \alpha) = D(p_1^0 - i\omega_e) D(p_1^0 - p_2^0 - i\alpha_5) D(p_2^0 - i\omega_e), \tag{77}$$

giving the following contribution to $I\{\omega_e\}$:

$$\int \frac{dp_1^0}{2\pi} \frac{dp_2^0}{2\pi} \sigma(p_1^0) \sigma(p_2^0) I_{12}(i\omega_e, p_1^0, p_2^0; \alpha). \tag{78}$$

The subdiagram corresponding to I_{12} is represented in Fig. 10. The expression (77) is well defined for all values of p_1^0, p_2^0, thanks to the regularization.

In this subsection, we have achieved our goal of expressing all the vacuum subdiagrams of the two-loop diagram of Fig. 5 in terms of analytic continuation of vacuum amplitudes (here 3-point and 4-point functions, or isolated propagators). Before going any further, we explain how this can be used in a simple analysis of ultraviolet divergences of Feynman diagrams at finite temperature.

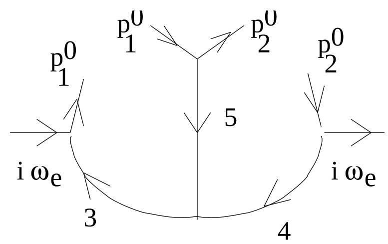


FIG. 10: Diagrammatic representation of the contribution $I_{12}(i\omega_e, p_1^0, p_2^0; \alpha)$ in Eq. (77).

C. Eliminating ultraviolet divergences in the two-loop example

We consider again the two-loop diagram of Fig. 5, calculated with the propagator D_0. It is of order g^4, where g is the coupling constant. We focus on the contributions involving an ultraviolet divergent subdiagram inside a finite temperature integral. These are the contributions with one thermal factor. By applying the rules of Sect. III, we obtain

$$I^{(1)}(i\omega_e; \alpha) = 2g^4 \int_{P_1} \sigma(p_1^0) I_1(i\omega_e, p_1^0; \alpha) + \frac{g^4}{2} \int_{P_5} \sigma(p_5^0) I_5(i\omega_e, p_5^0; \alpha), \tag{79}$$

where the notation $\int_P$ is for an integral over the four components of the momentum P. The factor 1/2 in the second term is the symmetry factor; the factor 2 in the first term arises from the 4 contributions identical to that of the first block in Fig. 11. Note that only the first integral contains ultraviolet divergences ($I_1(i\omega_e, p_1^0; \alpha)$ is ultraviolet divergent).

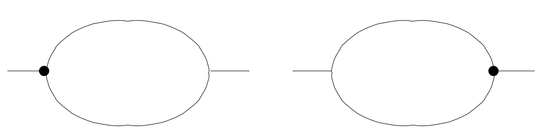


FIG. 11: Counter-terms diagrams; in each diagram, one vertex is associated to a coupling constant g, the other to the coupling constant counterterm δg.

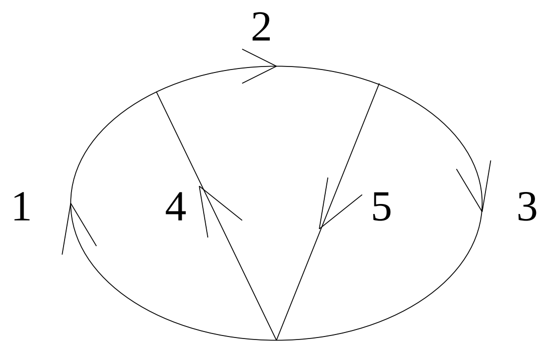


FIG. 12: Counter-example

We shall show that such temperature dependent, ultraviolet divergent, contributions are cancelled by the counterterm of order g^3 which eliminates the divergence in the one-loop contribution to the three-point vertex (see Fig. 8). Consider then the diagrams of Fig. 11 computed at finite temperature. We get:

$$I_{ct}(i\omega_e;\alpha) = 2g\delta g \int_{P_1} \sigma(p_1^0) D_0(p_1^0 + i\alpha_1 - i\omega_e), \tag{80}$$

where δg is the counterterm which eliminates the divergence of the diagram of Fig. 8. Combining this term with the first term of Eq. (79), one gets

$$2g \int_{P_1} \sigma(p_1^0) D_0(p_1^0 + i\alpha_1 - i\omega_e) \left[L(i\omega_e, p_1^0;\alpha) + \delta g\right]. \tag{81}$$

We argue now that $[L(i\omega_e, p_1^0;\alpha) + \delta g]$ is finite. This follows from the fact that $L(i\omega_e, p_1^0;\alpha)$ is the analytic continuation of the three-point vacuum amplitude $\Lambda(P_a, P_b)$ defined in Eq. (70), and the divergence of this amplitude is precisely that which is cancelled by the counterterm δg.

D. Counter-example

We now present an example of a diagram which contains vacuum subdiagrams that cannot be simply related to analytic continations of the topologically equivalent vacuum amplitudes. This diagram is displayed in Fig. 12.

We start by applying the rules of Sect. III and obtain:

$$I = \int_{12345} \hat{I}, \tag{82}$$

with

$$\hat{I} = -\frac{n_2 n_4 n_5 + n_2 n_{-1} n_{-3} + n_2 n_{-1} n_5 + n_2 n_4 n_{-3}}{(p_2^0 - p_1^0 - p_4^0)(p_3^0 + p_5^0 - p_2^0)} - \frac{n_3 n_5 n_4 + n_3 n_5 n_{-1}}{(p_1^0 + p_4^0 - p_3^0 - p_5^0)(p_3^0 + p_5^0 - p_2^0)} - \frac{n_1 n_4 n_5 + n_1 n_4 n_{-3}}{(p_1^0 + p_4^0 - p_3^0 - p_5^0)(p_2^0 - p_1^0 - p_4^0)}, \tag{83}$$

where we have used the shorthand notation introduced after Eq. (58).

Let us first verify that this formula has no infrared singularity associated with vanishing denominators. To this aim, let us set

$$A = p_2^0\,, \qquad B = p_3^0 + p_5^0\,, \qquad C = p_1^0 + p_4^0\,. \tag{84}$$

Then one can rewrite $\hat{I}$ as follows:

$$\hat{I} = (n_4 + n_{-1})(n_5 + n_{-3}) \left\{ \frac{n_A}{(A-B)(A-C)} + \frac{n_B}{(B-A)(B-C)} + \frac{n_C}{(C-A)(C-B)} \right\}, \tag{85}$$

where we have used relations such as $n_1 n_4 = n_{1+4}(n_4 + n_{-1})$. Consider now what happens when $A - B \to 0$ with $C \neq B$. Then, the first two terms in the brackets, wich are potentially divergent, lead in fact to a well defined limit:

$$\frac{n_A}{(A-B)(A-C)} + \frac{n_B}{(B-A)(B-C)} \to \frac{n'_B (B-C) - n_B}{(B-C)^2}, \tag{86}$$

where n'_B denotes the derivative of n_B with respect to B. If we further let $B - C \to 0$, we get

$$\frac{n'_B (B-C) - n_B + n_C}{(B-C)^2} \to \frac{1}{2} n''_C, \tag{87}$$

where n''_C is the second derivative of n_C with respect to C. Thus the limits where denominators vanish is well defined.

Now, in order to manipulate freely the various terms of Eq. (83), we introduce an infrared regularization by shifting p_j^0 by a small imaginary part, $p_j^0 \to p_j^0 + i\alpha_j$. The various α_j thus introduced are not all independent. Indeed, one may identify only three different factors in the denominators:

$$a = p_1^0 + p_4^0 - p_3^0 - p_5^0\,, \qquad b = p_2^0 - p_1^0 - p_4^0\,, \qquad c = p_3^0 + p_5^0 - p_2^0\,, \tag{88}$$

which furthermore satisfy the relation

$$a + b + c = 0. \tag{89}$$

Thus there are only two independent imaginary parts that we can play with. We shall attribute small imaginary parts to each of the factors a, b and c, respectively α_a, α_b and α_c, with the following constraints:

$$\alpha_a \neq 0\,,\ \alpha_b \neq 0\,,\ \alpha_c \neq 0, \qquad \alpha_a + \alpha_b + \alpha_c = 0, \tag{90}$$

from which it follows that:

$$\alpha_a + \alpha_b \neq 0, \qquad \alpha_b + \alpha_c \neq 0, \qquad \alpha_a + \alpha_c \neq 0. \tag{91}$$

The regularized expression of $\hat{I}$ in Eq. (83) reads then:

$$\begin{aligned}
\hat{I} \ = \ & -\frac{n_2 n_4 n_5 + n_2 n_{-1} n_{-3} + n_2 n_{-1} n_5 + n_2 n_4 n_{-3}}{(p_2^0 - p_1^0 - p_4^0 + i\alpha_b)(p_3^0 + p_5^0 - p_2^0 + i\alpha_c)} \\
& -\frac{n_3 n_5 n_4 + n_3 n_5 n_{-1}}{(p_1^0 + p_4^0 - p_3^0 - p_5^0 + i\alpha_a)(p_3^0 + p_5^0 - p_2^0 + i\alpha_c)} \\
& -\frac{n_1 n_4 n_5 + n_1 n_4 n_{-3}}{(p_1^0 + p_4^0 - p_3^0 - p_5^0 + i\alpha_a)(p_2^0 - p_1^0 - p_4^0 + i\alpha_b)}.
\end{aligned} \tag{92}$$

It has a well defined limit when $\alpha \to 0$.

We now split each statistical factor into a vacuum and a thermal piece and proceed to the identification of the various subdiagrams. We can go through the same analysis as before, and identify vacuum amplitudes, except for the terms with one thermal factor. Consider in particular the contribution I_1, the contribution where the line 1 is a thermal line. It is given by

$$\begin{aligned}
\hat{I}_1(p_1^0;\alpha) \ = \ & -\frac{\theta(p_2^0)\theta(p_3^0) + \theta(-p_2^0)\theta(-p_5^0)}{(p_2^0 - p_1^0 - p_4^0 + i\alpha_b)(p_3^0 + p_5^0 - p_2^0 + i\alpha_c)} \\
& -\frac{\theta(-p_3^0)\theta(-p_5^0)}{(p_1^0 + p_4^0 - p_3^0 - p_5^0 + i\alpha_a)(p_3^0 + p_5^0 - p_2^0 + i\alpha_c)} \\
& -\frac{\theta(-p_4^0)\theta(-p_5^0) - \theta(-p_4^0)\theta(p_3^0)}{(p_1^0 + p_4^0 - p_3^0 - p_5^0 + i\alpha_a)(p_2^0 - p_1^0 - p_4^0 + i\alpha_b)}.
\end{aligned} \tag{93}$$

One sees that in the denominators p_1^0 appears in the combinations $p_1^0 + i\alpha_a$ or $p_1^0 - i\alpha_b$. If one whishes to regard the function as an analytic continuation of a 2-point function depending on a single variable, it is then necessary to have $\alpha_a = -\alpha_b$. But this is in contradiction with the constraints (91). The other integrals I_2, I_3, I_4, I_5 suffer from the same difficulty.

V. CONCLUSIONS

Gaudin's method to perform the sums over the Matsubara frequencies leads to a very simple scheme for calculating Feynman diagrams at finite temperature: Once one has identified all the trees in the diagram, one gets for each tree a contribution in the form of a fraction whose numerator and denominator are given by simple rules. This method enables one to establish general properties such as those treated in the appendix. It can also be easily implemented on a computer [10].

One of our main concerns in this paper was the identification of vacuum subdiagrams, and their possible relations to analytical continutations of Euclidean amplitudes. This is relevant in particular to the discussion of ultraviolet divergent contributions in calculations at finite temperature. Gaudin's formula is useful in this context as it leads automatically to expressions in which the arguments of the statistical factors are the frequencies attached to the lines of the diagram, rather than combinations of such frequencies. This makes it easy to separate, in the total contribution of the diagram, thermal parts (sets of lines corresponding to the statistical factors), and vacuum parts (the rest of the lines in the diagram). Since the temperature cuts the flow of momentum in the thermal lines, the ultraviolet behaviour arises from the vacuum parts (note that this is true for generic propagators, not only for the perturbative one; that is, this reasoning applies to the general discussion in [9]). In several cases of practical interest, we have been able to relate these vacuum parts to vacuum amplitudes. However for general diagrams this identification is not always possible, at least in the way we have followed. The difficulty arises from the necessity to introduce a regularization which gives a meaning to isolated terms in Gaudin's formula. For the regularization that we have studied, we have shown that it is not always possible to identify vacuum parts with vacuum amplitudes. (Note that this difficulty does not alter the general proof given in [9]; it only makes its practical implementation more difficult.)

Finally, we note that the generalization to theories other than scalar theories is straightforward, since all the information about the theory is encoded in the spectral function which does not need to be specified in most part of the analysis presented in this paper.

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APPENDIX A: PROOF OF THE CONJECTURE OF REF. [6]

In order to illustrate the power of Gaudin's technique to calculate sums over Matsubara frequencies, we give here a simple proof of the main part of the conjecture stated in [6]. This conjecture concerns the possibility to reconstruct the expression of a Feynman diagram at finite temperature, starting from its corresponding expression at zero temperature, in the imaginary time formalism. The authors of Ref. [6] express their conjecture in an algebraic way, using a "thermal operator". Here we shall only show how the algorithm underlying their result emerges naturally from the rules of Sect. III.

Let us consider first the one-loop example of section II. We have obtained its expression at finite temperature, Eq. (18), which we recall here for convenience:

$$I(i\omega_e,\mathbf{k}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\varepsilon_p}\frac{1}{2\varepsilon_q}\left\{(1+n_{\varepsilon_p}+n_{\varepsilon_q})\left(\frac{1}{i\omega_e+\varepsilon_p+\varepsilon_q}-\frac{1}{i\omega_e-\varepsilon_p-\varepsilon_q}\right)\right.$$
$$\left. + (n_{\varepsilon_p}-n_{\varepsilon_q})\left(\frac{1}{i\omega_e-\varepsilon_p+\varepsilon_q}-\frac{1}{i\omega_e+\varepsilon_p-\varepsilon_q}\right)\right\}, \tag{A1}$$

with $\mathbf{q}=\mathbf{k}-\mathbf{p}$. The vacuum result is obtained from (A1) by dropping the terms proportional to the statistical factors ($n_{\varepsilon_p}=0$ at zero temperature):

$$I^{(0)}(i\omega_e,\mathbf{k}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\varepsilon_p}\frac{1}{2\varepsilon_q}\left\{\frac{1}{i\omega_e+\varepsilon_p+\varepsilon_q}-\frac{1}{i\omega_e-\varepsilon_p-\varepsilon_q}\right\}. \tag{A2}$$

The finite temperature contribution, $I^{(1)}$, is the sum of terms proportional to the statistical factors. The authors of [6] propose a simple algorithm to reconstruct $I^{(1)}$ from $I^{(0)}$. For each energy ε_p or ε_q appearing in the denominators of $I^{(0)}$, one adds a term proportional to a statistical factor n_{ϵ_p} or n_{ϵ_q} multiplied by a sum of two energy denominators, one of which is the original denominator of $I^{(0)}$, the other being obtained from it through the replacement

$\varepsilon_p \to -\varepsilon_p$ or $\varepsilon_q \to -\varepsilon_q$:

$$\begin{aligned}
\frac{1}{i\omega_e + \varepsilon_p + \varepsilon_q} &\to n_{\epsilon_p} \left\{ \frac{1}{i\omega_e + \varepsilon_p + \varepsilon_q} + \frac{1}{i\omega_e - \varepsilon_p + \varepsilon_q} \right\} \\
&+ n_{\epsilon_q} \left\{ \frac{1}{i\omega_e + \varepsilon_p + \varepsilon_q} + \frac{1}{i\omega_e + \varepsilon_p - \varepsilon_q} \right\}, \\
\frac{1}{i\omega_e - \varepsilon_p - \varepsilon_q} &\to n_{\epsilon_p} \left\{ \frac{1}{i\omega_e - \varepsilon_p - \varepsilon_q} + \frac{1}{i\omega_e + \varepsilon_p - \varepsilon_q} \right\} \\
&+ n_{\epsilon_q} \left\{ \frac{1}{i\omega_e - \varepsilon_p - \varepsilon_q} + \frac{1}{i\omega_e - \varepsilon_p + \varepsilon_q} \right\}.
\end{aligned} \tag{A3}$$

This procedure emerges naturally if one writes I as follows:

$$I(i\omega_e, k) = \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \frac{dq_0}{2\pi} \rho_0(p_0)\rho_0(q_0) \frac{n_{p_0} - n_{-q_0}}{p_0 + q_0 - i\omega_e}, \tag{A4}$$

with the free spectral density given by:

$$\rho_0(p_0) = 2\pi\epsilon(p_0)\delta(p_0^2 - \varepsilon_p^2) = \frac{\pi}{\varepsilon_p} \left\{ \delta(p_0 - \varepsilon_p) - \delta(p_0 + \varepsilon_p) \right\}. \tag{A5}$$

Indeed each statistical factor n_{p_0} contains a vacuum part $-\theta(-p_0)$ which selects one of the two peaks in the spectral density, giving in both cases a positive contribution, and a thermal part $\epsilon_{p_0} n_{|p_0|}$ with which both peaks in the spectral density contribute an equal and positive amount.

This result is easily generalized, as we show now. Let us consider a general diagram of perturbation theory ($\rho = \rho_0$) for which the sum over Matsubara frequency leads to the following integral (see Eq. (61)):

$$I\left\{i\omega_e\right\} = \sum_{\mathcal{T}} \prod_{l \in \bar{\mathcal{T}}} \int \frac{dp_l^0}{(2\pi)} \rho(p_l^0) \epsilon_l n_{\epsilon_l p_l^0} \prod_{j \in \mathcal{T}} D(\Omega_j; \alpha), \tag{A6}$$

where α denotes the regulators (see Sect. IV A). We shall focus on the contribution of a given tree, denoted by $I(\mathcal{T}; \alpha)$. This contains a vacuum contribution obtained after replacing each line of $\bar{\mathcal{T}}$ by a vacuum line carrying a factor $-\epsilon_l \theta(-\epsilon_l p_l^0)$. We shall denote this contribution by $I^{(0)}(\mathcal{T}; \alpha)$:

$$I^{(0)}(\mathcal{T}; \alpha) = \prod_{l \in \bar{\mathcal{T}}} \int \frac{dp_l^0}{(2\pi)} \rho(p_l^0) \left\{-\epsilon_l \theta(\epsilon_l p_l^0)\right\} \prod_{j \in \mathcal{T}} D(\Omega_j; \alpha). \tag{A7}$$

The other contributions to $I(\mathcal{T}; \alpha)$ are obtained by replacing some of the vacumm lines in $I^{(0)}(\mathcal{T}; \alpha)$ by thermal lines carrying factors $\varepsilon(p_l^0) n_{|p_l^0|}$. The contribution for which the subset $\mathcal{S}$ of lines of $\bar{\mathcal{T}}$ are thermal lines reads:

$$I_{\mathcal{S}}(\mathcal{T}; \alpha) = \int \prod_{\bar{l} \in \bar{\mathcal{T}}} \frac{dp_{\bar{l}}^0}{(2\pi)} \rho(p_{\bar{l}}^0) \prod_{l' \in \bar{\mathcal{S}}} \left\{-\epsilon_{l'} \theta(-\epsilon_{l'} p_{l'}^0)\right\} \prod_{l \in \mathcal{S}} \epsilon_l n_{\epsilon_l p_l^0} \prod_{j \in \mathcal{T}} D(\Omega_j; \alpha), \tag{A8}$$

where $\bar{\mathcal{S}}$ denotes the set of lines of $\bar{\mathcal{T}}$ which remain vacuum lines.

At this point we can repeat the same argument as in the one-loop example above. In $I^{(0)}(\mathcal{T};\alpha)$, when we integrate over the free spectral densities, each θ-function selects one of the peaks in the spectral density and gives always a positive contribution, whatever the selected peak is. The denominators are determined by plugging in the Ω_j's the energies ε_p corresponding to the (selected) peaks of the spectral functions, with the signs given by the rules of Sect. III. In going from $I^{(0)}(\mathcal{T};\alpha)$ to $I_{\mathcal{S}}(\mathcal{T};\alpha)$, we replace some of θ-functions by contributions that are proportional to a thermal factor and in which the two peaks in the spectral density contribute on the same footing, leading to a duplication of denominators with the values $\pm\varepsilon_p$ of the energies. This is essentially the content of "Statement 1" in [6], the thermal operator introduced there being the operator relating $I_{\mathcal{S}}$ in Eq. (A8) to $I^{(0)}$ in Eq. (A7).

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